Name KEY Date________________________ Period______

Worksheet 2.3—Differentiation Rules

Show all work. No Calculator unless stated otherwise.

Short Answer

For 1-3, using correct notation (always), find the derivatives of the following functions. Simplify early and often!! Be sure to consider rewriting each term in the correct form first.

1. $f(x) = -2x^3 - x^2 + 4x - 7$

$f'(x) = -6x^2 - 2x + 4$

2. $g(x) = \dfrac{1}{x} - 3\sin x + \dfrac{4x + 2\sqrt{x}}{3\sqrt[3]{x}}$

$g(x) = x^{-1} - 3\sin x + \frac{4}{3}x^{1-1/3} + \frac{2}{3}x^{1/2-1/3}$

$g(x) = x^{-1} - 3\sin x + \frac{4}{3}x^{2/3} + \frac{2}{3}x^{1/6}$

$g'(x) = -x^{-2} - 3\cos x + \frac{8}{9}x^{-1/3} + \frac{1}{9}x^{-5/6}$

$g'(x) = -\frac{1}{x^2} - 3\cos x + \frac{8}{9\sqrt[3]{x}} + \frac{1}{9\sqrt[6]{x^5}}$

3. $y = 4x^2(3-2x)^2$

$y = 4x^2(9 - 12x + 4x^2)$

$y = 36x^2 - 48x^3 + 16x^4$

$y' = \frac{dy}{dx} = 72x - 144x^2 + 64x^3$

For 4-6, using correct notation (never not always), evaluate each of the following with respect to the indicated variable. Simplify early and often!! Be sure to consider rewriting each term in the correct form first.

4. $\dfrac{d}{dt}\left[\sqrt{3t} - 6\sqrt[4]{t} - 4\cos t - \pi\right] =$

$\frac{d}{dt}\left[\sqrt{3}t^{1/2} - 6t^{1/4} - 4\cos t - \pi\right]$

$\frac{\sqrt{3}}{2}t^{-1/2} - \frac{6}{4}t^{-3/4} + 4\sin t$

$\frac{\sqrt{3}}{2\sqrt{t}} - \frac{3}{2\sqrt[4]{t^3}} + 4\sin t$

5. $\dfrac{d}{dx}\left[\left(\dfrac{x}{x^2+1}\right)^{-1}\right] =$

$\frac{d}{dx}\left[\frac{x^2+1}{x}\right]$

$\frac{d}{dx}\left[\frac{x^2}{x} + \frac{1}{x}\right]$

$\frac{d}{dx}\left[x + x^{-1}\right]$

$1 - x^{-2}$

$1 - \frac{1}{x^2}$ or $\frac{x^2-1}{x^2}$

6. $\dfrac{d}{dm}\left[\dfrac{m^{-1} + m^{-2}}{m^{-3}}\right] =$

$\frac{d}{dm}\left[\frac{\frac{1}{m} + \frac{1}{m^2}}{\frac{1}{m^3}}\left(\frac{m^3}{m^3}\right)\right]$

$\frac{d}{dm}\left[\frac{m^2 + m}{1}\right]$

$\frac{d}{dm}\left[m^2 + m\right]$

$2m + 1$

7. For $f(x)=(x^2+2x)(x+1)$, find $f'(x)$. Remember to simplify early and often!

Expand:
$f(x)=x^3+x^2+2x^2+2x$
$f(x)=x^3+3x^2+2x$
$f'(x)=3x^2+6x+2$

(a) find the equation of the tangent line, in Taylor form, to the graph of f at $x=1$.

pt: $(1, f(1))$ $(1, 6)$ m: $f'(1)=11$ equation $y=6+11(x-1)$

(b) find the equation of the normal line, in Taylor form, to the graph of f at $x=1$.

pt: $(1,6)$ $m=\frac{-1}{f'(1)}=\frac{-1}{11}$ equation $y=6-\frac{1}{11}(x-1)$

(c) Using your equation from part (b), find the x-intercept of the normal line. Show the work that leads to your answer.

x-int: $y=0$
$0=6-\frac{1}{11}(x-1)$
$\frac{1}{11}(x-1)=6$
$x-1=66$
$x=67$
x-int: $(67, 0)$

For 8-10, determine the point(s) (if any) at which the graph of the following functions have horizontal tangent lines. Justify.

8. $y=x+\sin x,\ 0\le x<2\pi$

$y'=1+\cos x$
$1+\cos x=0$
$\cos x=-1$
$x=\pi$
y has a horizontal tangent line at $x=\pi$. (or at (π,π))

9. $y=\sqrt{3}x+2\cos x,\ x\in[0,2\pi)$

$y'=\frac{dy}{dx}=\sqrt{3}-2\sin x=0$
$-2\sin x=-\sqrt{3}$
$\sin x=\frac{\sqrt{3}}{2}$
$x=\frac{\pi}{3}, \frac{2\pi}{3}$

10. $f(x)=\frac{2}{x^2}$

$f(x)=2x^{-2}$
$f'(x)=-4x^{-3}$
$f'(x)=\frac{-4}{x^3}\neq 0$
So f(x) has No horizontal tangent lines

For 11 & 12, find the value of k such that the given line is tangent to the graph of the given function.

11. $f(x)=x^2-kx$, line $y=4x-9$

At any point of tangency, a function and its tangent line share both a ① y-value ② slope value

y-values: $f(x)=y$, $x^2-kx=4x-9$

slopes: $f'(x)=y'$, $2x-k=4$, So, $k=2x-4$

Subbing into y eq.

$x^2-(2x-4)x=4x-9$

$x^2-2x^2+4x=4x-9$

$-x^2=-9$

$x=3$ or $x=-3$

So, $k=2(3)-4$ or $k=2(-3)-4$

$k=2$ $\quad k=-10$

* 2 solutions

12. $f(x)=k\sqrt{x}$, line $y=x+4$

$f(x)=kx^{1/2}$

$f'(x)=\frac{k}{2}x^{-1/2}$

$f'(x)=\frac{k}{2\sqrt{x}}$

$y=mx+b$

$y=x+4$

$\boxed{m=1}$

* At any point of tangency, the function and its tangent line share two things: ① A y-value ② A slope value

y-value: $k\sqrt{x}=x+4$, $k=\frac{x+4}{\sqrt{x}}$

slope value: $\frac{k}{2\sqrt{x}}=1$, $k=2\sqrt{x}$

So $\frac{x+4}{\sqrt{x}}=2\sqrt{x}$

$x+4=2\sqrt{x}\cdot\sqrt{x}$

$x+4=2x$

$4=x$

$\boxed{x=4}$

So $k=2\sqrt{4}$

$\boxed{k=4}$

Questions 13-16 are True of False. If False, either explain why, rewrite it to make it true, or provide a counterexample.

13. If $f'(x)=g'(x)$, then $f(x)=g(x)$.

False, f and g can differ by a constant.

for example

$f(x)=x^2+1$

$g(x)=x^2+3$

$f'(x)=2x=g'(x)$, but $f(x)\neq g(x)$

14. If $f(x)=g(x)+C$, then $f'(x)=g'(x)$.

True

$\frac{d}{dx}[f(x)]=\frac{d}{dx}[g(x)+C]$

$\frac{d}{dx}[f(x)]=\frac{d}{dx}[g(x)]+\frac{d}{dx}[C]$

$f'(x)=g'(x)+0$

$f'(x)=g'(x)$

15. If $y=\pi^3$, then $\frac{dy}{dx}=3\pi^2$.

False, π^3 is a constant

So, $\frac{d}{dx}[\pi^3]=0$

16. If $f(x)=\frac{1}{x^n}$, then $f'(x)=\frac{1}{nx^{n-1}}$

False

$f(x)=x^{-n}$

$f'(x)=-nx^{-n-1}$

$f'(x)=-n\cdot x^{-(n+1)}$

$=\frac{-n}{x^{n+1}}\neq\frac{1}{nx^{n-1}}$

17. (Calculator permitted) A priceless Faberge egg is dropped from the top of a building that is 1362 feet tall. The egg's height in feet at time t seconds is given by $h(t) = -16t^2 + 1362$.

(a) Find the average velocity, in ft/sec, of the egg on the interval $[1, 2]$ seconds. Show the work that leads to your answer (always).

$$\text{Avg. vel.} = \frac{h(2) - h(1)}{2-1}$$
$$= \frac{((-16)(4) + 1362) - (-16 + 1362)}{1}$$
$$= -64 + 16$$
$$= -48 \text{ ft/sec}$$

(b) Based on your answer in part (a), fill in the blanks so that the following sentence verbally describes the result found above.

"On the interval from $t =$ 1 seconds to $t =$ 2 seconds, the egg's height is decreasing (increasing/decreasing), on average, by 48 feet per second."

(c) Find the instantaneous velocity of the egg at $t = 2$ seconds. Show the work that leads to your answer (always).

$h'(t) = v(t) = -32t$

$V(2) = -32(2)$
$= -64 \text{ ft/sec}$

(d) Based on your answer in part (c), fill in the blanks so that the following sentence verbally describes the result found above.

"At $t = 2$ seconds, the egg's height is decreasing (increasing/decreasing), by 64 feet per second."

(e) After how many seconds will the egg hit the ground? Show the work that leads to your answer.

$h(t) = 0$
$-16t^2 + 1362 = 0$
$16t^2 = 1362$
$t = \pm\sqrt{\frac{1362}{16}}$
$t = 9.226$ seconds $= A$ (store as "A")

(f) Find the velocity, in ft/sec, of the egg as it hits the ground.

$V(9.226)$
$V(A)$
-295.242 ft/sec

18. Find the values of a and/or b ($a \neq 0$), if they exist, such that f is differentiable for all x. As always, show the work that leads to your answer.

(a) $f(x)=\begin{cases} ax^3, & x \le 2 \\ x^2+b, & x>2 \end{cases}$

Continuity

$\lim_{x\to 2^-} f(x) = f(2) = 8a$

$\lim_{x\to 2^+} f(x) = 4+b$

So, $8a = 4+b$

So, $8\left(\frac{1}{3}\right) = 4+b$

$b = \frac{8}{3} - 4$

$\boxed{b = -\frac{4}{3}}$

derivatives

$f'(x) = \begin{cases} 3ax^2, & x<2 \\ 2x, & x>2 \end{cases}$

$\lim_{x\to 2^-} f'(x) = 3a(2^2) = 12a$

$\lim_{x\to 2^+} f'(x) = 2(2) = 4$

So, $12a = 4$

$a = \frac{4}{12}$

$\boxed{a = \frac{1}{3}}$

(b) $f(x)=\begin{cases} \cos x, & x < \dfrac{\pi}{2} \\ ax+b, & x \ge \dfrac{\pi}{2} \end{cases}$

$f' = \begin{cases} -\sin x, & x < \frac{\pi}{2} \\ a, & x \geq \frac{\pi}{2} \end{cases}$

Continuity

$\lim_{x\to \pi/2^-} f(x) = \cos\frac{\pi}{2} = 0$

$\lim_{x\to \pi/2^+} f(x) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}a + b$

So (1) $\frac{\pi}{2}a + b = 0$

Slopes

$\lim_{x\to \frac{\pi}{2}^-} f'(x) = -\sin\frac{\pi}{2} = -1$

$\lim_{x\to \frac{\pi}{2}^+} f'(x) = f'\left(\frac{\pi}{2}\right) = a$

(2) So $a = -1$

when $a = -1$ (2)→(1)

$\frac{\pi}{2}(-1) + b = 0$

$\boxed{b = \frac{\pi}{2}}$

So $a = -1$ & $b = \frac{\pi}{2}$

(c) $f(x)=\begin{cases} \sin x, & x \le 0 \\ ax, & x>0 \end{cases}$

Continuity

$\lim_{x\to 0^-} f(x) = f(0) = \sin 0 = 0$

$\lim_{x\to 0^+} f(x) = 0(a) = 0$

$0 = 0$,

so, $f(x)$ is always continuous at $x=0$ for all a values.

derivative

$f'(x) = \begin{cases} \cos x, & x<0 \\ a, & x>0 \end{cases}$

$\lim_{x\to 0^-} f'(x) = \cos 0 = 1$

$\lim_{x\to 0^+} f'(x) = a$

So, $\boxed{a=1}$

(d) $f(x)=\begin{cases} ax^2, & x \le 1 \\ b\sqrt{x}, & x>1 \end{cases}$

Continuity

$\lim_{x\to 1^-} f(x) = f(1) = a(1^2) = a$

$\lim_{x\to 1^+} f(x) = b\sqrt{1} = b$

So, $a = b$

So $a = 4a$

$1 = 4$

$\Rightarrow\Leftarrow$

contradiction

derivative

$f'(x) = \begin{cases} 2ax, & x<1 \\ \dfrac{b}{2\sqrt{x}}, & x>1 \end{cases}$

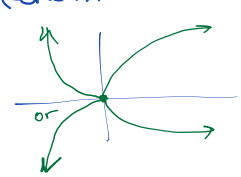
$\lim_{x\to 1^-} f'(x) = 2a(1) = 2a$

$\lim_{x\to 1^+} f'(x) = \frac{b}{2\sqrt{1}} = \frac{b}{2}$

So, $2a = \frac{b}{2}$

$b = 4a$

So, no such values of a & b exist to make $f(x)$ differentiable at $x = 1$.



* The graph of $f(x)$ will be either branch left of zero plus either branch right of zero. Any one of the 4 possible graphs will have a CUSP at $x=0$.

Multiple Choice

D 19. The function $f(x) = 3\sqrt[3]{x^2} + x - 1$ is differentiable for which values of x?

(A) all real numbers (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) for all $x \neq 0$ (E) $(-\infty, 0)$

$f(x) = 3x^{2/3} + x - 1$, $f(x)$ is continuous for all x.

cusp alert @ $x=0$!!

$f'(x) = 2x^{-1/3} + 1$

$f'(x) = \frac{2}{\sqrt[3]{x}} + 1$

$D_{f'}: \{x \mid x \neq 0\}$

So $f(x)$ is differentiable for all $x \neq 0$.

E 20. The function $f(x) = |x+4| - \sqrt[5]{x^3} + \frac{1}{x-3}$ is differentiable for all x-values except

Sharp turn at $x=-4$

vertical tangent at $x=0$

VA (discontinuity) at $x=3$

I. $x = -4$
II. $x = 0$
III. $x = 3$

(A) I only (B) I and II only (C) I and III only (D) III only (E) I, II, and III

C 21. On the interval $[0, 2\pi)$, for which of the following x-values is the function $f(x) = \tan x$ not differentiable?

(A) all real numbers (B) $x = \frac{\pi}{4}$ (C) $x = \frac{3\pi}{2}$ (D) $x = \pi$ (E) $x = 0$

$y = \tan x$ has vertical asymptotes at odd $\pi/2$